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A new approach to formulate structural methods for multibody dynamics

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ABSTRACT

In this paper we present a reformulation of the so called structural methods for the integration of multibody dynamics. This reformulation allows one to overcome some numerical issues that appear in some of the structural methods when they are applied to multibody dynamics. It also allows for the development of a generalization of these methods, which leads to new algorithms of a higher order of convergence.

The reformulation leads to three new families of methods, which can be classified as explicit, conditionally explicit and implicit. All of them can be directly applied to second order Ordinary Differential Equations (EDOs) and allow to introduce restrictions directly in terms of the function to integrate, thus not requiring stabilization or projection methods.

In the case of Differential Algebraic Equations (DAEs) one can have stability problems derived from the restrictions. This do not appear in the case of the explicit and the conditionally explicit methods, but they are still present in the implicit methods In any case, these problems can be dealt with in the same methods that are usually applied to Newmark.

In the case of implicit methods, the obtained formulation is of a single point scheme and in the case of explicit and conditionally explicit methods, the obtained formulation is similar to a single point scheme, but it takes into account values from two timesteps, although not all of them.

The methods are first formulated, afterwards, some examples are presented where some hints on the performance of the methods can be derived.

Keywords: Structural Integrators, Multibody Dynamics, Newmark, Central Differences.

1 INTRODUCTION

The use of the so called structural integrators is becoming a mainstream in Multibody Dynamics [1]–[3]. These methods allow one to directly integrate the Differential Algebraic Equations (DAEs) that appear in Multibody Dynamics and other phenomena without the need of reducing the DAE index, although this can lead in the case of implicit methods to difficult to predict stability issues, which can be avoided by integrating in minimal coordinates [4].

Currently the most used implicit structural integrators in multibody dynamics are the Newmark method and the HHT method. Recently, the second order central difference method (SOCDM) has also succesfully being introduced. This method is considered as explicit for its use in structural dynamics, but in the case of 3D Multibody Dynamics it becomes implicit due to the dependency on the velocity that appears in the differential equation. Probably the biggest drawback of these methods is that they are limited to order 2 convergence. The classical formulation of SOCDM used in structural dynamics leads also to a cancellation problem that reduces its precision.

In this document the authors will present a reformulation of both the Newmark and the SOCDM methods, which allow one to overcome the problems of cancellation in the classical SOCDM and leads to a configurable and extensible formulation of the methods. The extension of these reformulations leads to integrators which exhibit higher convergence than their classical counterpart (obviously at the cost of reduction of the stability conditions).

2 CENTRAL DIFFERENCES AND NEWMARK METHODS IN STRUCTURAL DYNAMICS AND MULTIBODY DYNAMICS

The most used methods to integrate structural dynamics in Finite Element models are the Newmark method (implicit) and the Central Differences method (considered explicit) [5]. The central differences method in structural dynamics is different from the classical Runge Kutta Central Differences. In structural dynamics, it employs two equations to relate velocity and acceleration with deformation.

$$\dot{\delta}(t)=\frac{x(t+\Delta t)-x(t-\Delta t)}{2\Delta t},\ddot{\delta}(t)=\frac{x(t+\Delta t)-2x(t)+x(t-\Delta t)}{\Delta t^2} \tag{1}$$

which, along with the equilibrium equation evaluated in the beginning of the timestep:

$$M(t)\ddot{\delta}(t)+C(t)\dot{\delta}(t)+K(t)\delta(t)=f(t) \tag{2}$$

conform the integrator. This integrator allow one to obtain $\delta(t+\Delta t)$, $\dot{\delta}(t)$ and $\ddot{\delta}(t)$ from $\delta(t)$, $\dot{\delta}(t-\Delta t)$ and $\ddot{\delta}(t-\Delta t)$. For structural dynamics this is usually reformulated leading to a multipoint formulation, leading to an equation in the form:

$$\left(\frac{M}{(\Delta t)^2}+\frac{C}{2\Delta t}\right)\delta(t+\Delta t)=f(t)+\left(\frac{-M}{(\Delta t)^2}+\frac{C}{2\Delta t}\right)\delta(t-\Delta t)-\left(K-\frac{2M}{(\Delta t)^2}\right)\delta(t) \tag{3}$$

Which allows one to obtain $\delta(t+\Delta t)$ from $\delta(t)$ and $\delta(t-\Delta t)$. For structural dynamics, the method is explicit (although the method itself is not explicit, more on this later). It has second order accuracy and is conditionally stable, which means that there is a limit on the timestep required for the method to be stable.

The Newmark method in structural dynamics uses the equations:

$$\delta(t+\Delta t)=\delta(t)+\Delta t\cdot\dot{\delta}(t)+\frac{(\Delta t)^2}{2}\left((1-\alpha_0)\ddot{\delta}(t)+\alpha_0\ddot{\delta}(t+\Delta t)\right) \tag{4}$$

and:

$$\dot{\delta}(t+\Delta t)=\dot{\delta}(t)+\Delta t\left((1-\alpha_1)\ddot{\delta}(t)+\alpha_1\ddot{\delta}(t+\Delta t)\right) \tag{5}$$

In this case, the equilibrium equation is formulated in $t+\Delta t$:

$$M(t+\Delta t)\ddot{\delta}(t+\Delta t)+C(t+\Delta t)\dot{\delta}(t+\Delta t)+K(t+\Delta t)\delta(t+\Delta t)=f(t+\Delta t) \tag{6}$$

It is a configurable method, whose properties vary depending on the values taken for α_0 and α_1 (usually instead of α_0 and α_1, the teminology $\alpha=\alpha_1$and $\beta=\alpha_0/2$). The method is second order convergent if $\alpha_1=0.5$ and it is stable provided that $\alpha_0\geq\alpha_1$ and $\alpha_1>0.5$ (for single point formulations).

If one is to use these methods in Multibody Dynamics, one only needs to replace deformations by positions and introduce a proper 3D orientation system along with the restrictions that it might require.

The use of Newmark has not been uncommon in Multibody Dynamics, and it has several advantages over Runge Kutta formulations. It shares with R-K methods the problematic of the stability when the equations to integrate are DAEs. This has traditionally been solved by introducing large values for α_0 and α_1. But this has a couple of drawbacks. First, one introduces a considerable amount of numerical damping and, second, the convergence order is reduced. The last issue has been successfully solved by introducing the HHT method, but it still introduces a considerable amount of numerical damping (although it usually increases with the natural frequency, which is a good feature). Another possibility is the use of a minimal coordinate set. This is usually performed in periods of several timesteps, but one can also take advantage of the use of nullspaces to perform the reduction of coordinates in each timestep at a reduced cost.

In the other hand, the use of the Central Difference Method has not been quite common. This is probably due to the fact that, while explicit in Structural Dynamics, it is implicit in 3D Multibody Dynamics, due to the dependence of C on $\dot{x}$. This is the reason why, in this document, we shall classify it as “conditionally explicit”. However, some recent experiments with it show that it can be a good performer.

3 REFORMULATING CENTRAL DIFFERENCES

As stated before, the central differences method for structural dynamics has usually been formulated as a multipoint method, in a similar way to ADAMS methods. This has a drawback, namely the fact that velocities can only be obtained in a finite difference approach. This is quite inconvenient, because it leads to heavy cancellation, thus limiting the achievable precision. It is easy to find out that one can reformulate the method by using the equations:

$$x(t+\Delta t)=x(t)+\Delta t\,\dot{x}(t)+\frac{\Delta t^2}{2}\ddot{x}(t-\Delta t)+\Delta t^2\ddot{x}(t) \tag{7}$$

$$\dot{x}(t)=\dot{x}(t-\Delta t)+\frac{\Delta t}{2}\ddot{x}(t-\Delta t)+\frac{\Delta t}{2}\ddot{x}(t) \tag{8}$$

Using these equations along with the equilibrium equation allow one to obtain $\ddot{x}(t)$, $\dot{x}(t)$ and $x(t+\Delta t)$ from $\ddot{x}(t-\Delta t)$, $\dot{x}(t-\Delta t)$ and $x(t)$. As one obtains $\dot{x}(t)$ in each timestep, the central difference expression is not required and, thus, this cancellation problem dissapears.

One can easily find a similitude with this formulation of the Central Differences method and Newmarks method. This leads to the possibility of developing a configurable method for central differences. By introducing a set of α_0, α_1 parameters, one can reach:

$$x(t+\Delta t)=x(t)+\Delta t\,\dot{x}(t)+\frac{\Delta t^2}{4}(1-\alpha_0)\ddot{x}(t-\Delta t)+\alpha_0\frac{\Delta t^2}{4}\ddot{x}(t) \tag{9}$$

$$\dot{x}(t)=\dot{x}(t-\Delta t)+\frac{\Delta t}{2}(1-\alpha_1)\ddot{x}(t-\Delta t)+\frac{\Delta t}{2}\alpha_1\ddot{x}(t) \tag{10}$$

Which leads to a configurable method.

4 A FAMILY OF EXPLICIT METHODS

Let us now consider Newmark equations and the new reformulation of central differences. One could also consider using a set of equations in the form:

$$x(t+\Delta t)=x(t)+\Delta t\,\dot{x}(t)+\frac{\Delta t^2}{2}\alpha_0\ddot{x}(t)+\frac{\Delta t^2}{2}(1-\alpha_0)\ddot{x}(t-\Delta t) \tag{11}$$

$$\dot{x}(t+\Delta t)=\dot{x}(t)+\Delta t\,\alpha_1\ddot{x}(t)+\Delta t(1-\alpha_1)\ddot{x}(t-\Delta t) \tag{12}$$

Which, along with the equilibrium equation evaluated in t, conforms an explicit integrator. It would allow one to obtain $x(t+\Delta t)$, $\dot{x}(t+\Delta t)$ and $\ddot{x}(t)$ from $x(t)$, $\dot{x}(t)$ and $\ddot{x}(t-\Delta t)$. Unfortunately, this method is unstable. But one might take into account that all equations that lead to the here presented methods can be considered as weighted variations of a Taylor series. In fact, if one introduces $\alpha_0=\alpha_1=1$, Taylor series expressions are obtained. One might consider increasing the amount of terms considered in the Taylor series. For example, one can use:

$$x(t+\Delta t)=x(t)+\Delta t\,\dot{x}(t)+\frac{\Delta t^2}{2}\ddot{x}(t)+\frac{\Delta t^3}{6}\alpha_0\dddot{x}(t)+\frac{\Delta t^3}{6}(1-\alpha_0)\dddot{x}(t-\Delta t) \tag{13}$$

$$\dot{x}(t+\Delta t)=\dot{x}(t)+\Delta t\,\ddot{x}(t)+\frac{\Delta t^2}{2}\alpha_1\dddot{x}(t)+\frac{\Delta t^2}{2}(1-\alpha_1)\dddot{x}(t-\Delta t) \tag{14}$$

but one needs the equilibrium equation and an additional equation. Although it looks interesting to take a derivative on the equilibrium equation, that would lead to a complicated method and no advantage at all. Instead, it is better to introduce:

$$\ddot{x}(t)=\ddot{x}(t-\Delta t)+\Delta t\,\alpha_2\dddot{x}(t)+\Delta t(1-\alpha_2)\dddot{x}(t-\Delta t) \tag{15}$$

The method allows one to obtain $x(t+\Delta t)$, $\dot{x}(t+\Delta t)$, $\ddot{x}(t)$ and $x^{(3)}(t)$ from $x(t)$, $\dot{x}(t)$, $\ddot{x}(t-\Delta t)$ and $x^{(3)}(t-\Delta t)$. In this case, there are some sets of parameters that render the method conditionally stable. For example, the use of $\alpha_0=5/4$, $\alpha_1=4/3$ and $\alpha_2=1/2$. Furthermore, with these parameters it is third order convergent.

An even more general expression can be arranged. By using equations (16)-(19), along with the equilibrium equation at the beginning of the timestep (20):

$$x(t+\Delta t)=\sum_{k=0}^{g-2}\frac{\Delta t^k}{k!}\left|\frac{d^k x}{dt^k}\right|_t+\frac{\Delta t^{g-1}}{(g-1)!}\left(\alpha_0\left|\frac{d^{(g-1)}x}{dt^{g-1}}\right|_t(1-\alpha_0)\left|\frac{d^{(g-1)}x}{dt^{g-1}}\right|_{t-\Delta t}\right) \tag{16}$$

$$\dot{x}(t+\Delta t)=\sum_{k=1}^{g-2}\frac{\Delta t^{k-1}}{(k-1)!}\left|\frac{d^k x}{dt^k}\right|_t+\frac{\Delta t^{g-2}}{(g-2)!}\left(\alpha_1\left|\frac{d^{(g-1)}x}{dt^{g-1}}\right|_t(1-\alpha_1)\left|\frac{d^{(g-1)}x}{dt^{g-1}}\right|_{t-\Delta t}\right) \tag{17}$$

$$\ddot{x}(t)=\sum_{k=2}^{g-2}\frac{\Delta t^{k-2}}{(k-2)!}\left|\frac{d^k x}{dt^k}\right|_{t-\Delta t}+\frac{\Delta t^{g-3}}{(g-3)!}\left((1-\alpha_2)\left|\frac{d^{(g-1)}x}{dt^{g-1}}\right|_{t-\Delta t}+\alpha_2\left|\frac{d^{(g-1)}x}{dt^{g-1}}\right|_t\right) \tag{18}$$

…

$$\left|\frac{d^{g-2}x}{dt^{g-2}}\right|_t=\left|\frac{d^{g-2}x}{dt^{g-2}}\right|_{t-\Delta t}+\Delta t\left((1-\alpha_{g-2})\left|\frac{d^{g-1}x}{dt^{g-1}}\right|_{t-\Delta t}+\alpha_{g-2}\left|\frac{d^{g-1}x}{dt^{g-1}}\right|_t\right) \tag{19}$$

$$\ddot{\boldsymbol{x}}(t)=\boldsymbol{f}(\boldsymbol{x}(t),\dot{\boldsymbol{x}}(t),t) \tag{20}$$

One reaches a general g-degree method which can be considered as a family of explicit methods. We use the term degree to refer to the amount of derivatives to be taken into account in the process (including the function itself).

5 A FAMILY OF CONDITIONALLY EXPLICIT METHODS

We now go back to the central difference method. By applying the same considerations exposed for explicit methods, one can consider using equations (21)-(24)along with the equilibrium equation at the beginning of the timestep (25) to conform a family of conditionally explicit methods:

$$x(t+\Delta t)=\sum_{k=0}^{g-2}\frac{\Delta t^k}{k!}\left|\frac{d^k x}{dt^k}\right|_t+\frac{\Delta t^{g-1}}{(g-1)!}\left((1-\alpha_0)\left|\frac{d^{g-1}x}{dt^{g-1}}\right|_{(t-\Delta t)}+\alpha_0\left|\frac{d^{g-1}x}{dt^{g-1}}\right|_t\right) \tag{21}$$

$$\dot{x}(t)=\sum_{k=1}^{g-2}\frac{\Delta t^{k-1}}{(k-1)!}\left|\frac{d^k x}{dt^k}\right|_{t-\Delta t}+\frac{\Delta t^{g-2}}{(g-2)!}\left((1-\alpha_1)\left|\frac{d^{g-1}x}{dt^{g-1}}\right|_{(t-\Delta t)}+\alpha_1\left|\frac{d^{g-1}x}{dt^{g-1}}\right|_t\right) \tag{22}$$

$$\ddot{x}(t)=\sum_{k=2}^{g-2}\frac{\Delta t^{k-2}}{(k-2)!}\left|\frac{d^k x}{dt^k}\right|_{t-\Delta t}+\frac{\Delta t^{g-3}}{(g-3)!}\left((1-\alpha_2)\left|\frac{d^{g-1}x}{dt^{g-1}}\right|_{(t-\Delta t)}+\alpha_2\left|\frac{d^{g-1}x}{dt^{g-1}}\right|_t\right) \tag{23}$$

…

$$\left|\frac{d^{g-2}x}{dt^{g-2}}\right|_{t}=\left|\frac{d^{g-2}x}{dt^{g-2}}\right|_{t-\Delta t}+\Delta t\left(\left(1-\alpha_{g-2}\right)\left|\frac{d^{g-1}x}{dt^{g-1}}\right|_{(t-\Delta t)}+\alpha_{g-2}\left|\frac{d^{g-1}x}{dt^{g-1}}\right|_{t}\right) \tag{24}$$

$$\ddot{\boldsymbol{x}}(t)=\boldsymbol{f}(\boldsymbol{x}(t),\dot{\boldsymbol{x}}(t),t) \tag{25}$$

Until now, we have found conditionally stable configurations of the method up until 5th degree. We have introduced the terminology "conditionally explicit methods" because these methods are implicit in nature, but, if (25) is linear in $\dot{\boldsymbol{x}}$, it can be expressed in terms of, for example, acceleration by using (22) and, thus, it can be removed from the equation, thus leading to a explicit integrator.

6 A FAMILY OF IMPLICIT METHODS

Obviously, one can also extend Newmark method. By using the equations (26)-(29) along with the equilibrium equation (30) (this time evaluated at the end of the timestep), a family of implicit methods is conformed:

$$x(t+\Delta t)=\sum_{k=0}^{g-2}\frac{\Delta t^{k}}{k!}\left|\frac{d^{k}x}{dt^{k}}\right|_{t}+\frac{\Delta t^{g-1}}{(g-1)!}\left(\alpha_{0}\left|\frac{d^{(g-1)}x}{dt^{g-1}}\right|_{t+\Delta t}+\left(1-\alpha_{0}\right)\left|\frac{d^{(g-1)}x}{dt^{g-1}}\right|_{t}\right) \tag{26}$$

$$\dot{x}(t+\Delta t)=\sum_{k=1}^{g-2}\frac{\Delta t^{k-1}}{(k-1)!}\left|\frac{d^{k}x}{dt^{k}}\right|_{t}+\frac{\Delta t^{g-2}}{(g-2)!}\left(\alpha_{1}\left|\frac{d^{(g-1)}x}{dt^{g-1}}\right|_{t+\Delta t}+\left(1-\alpha_{1}\right)\left|\frac{d^{(g-1)}x}{dt^{g-1}}\right|_{t}\right) \tag{27}$$

$$\ddot{x}(t+\Delta t)=\sum_{k=2}^{g-2}\frac{\Delta t^{k-2}}{(k-2)!}\left|\frac{d^{k}x}{dt^{k}}\right|_{t}+\frac{\Delta t^{g-3}}{(g-3)!}\left(\left(1-\alpha_{2}\right)\left|\frac{d^{(g-1)}x}{dt^{g-1}}\right|_{t}+\alpha_{2}\left|\frac{d^{(g-1)}x}{dt^{g-1}}\right|_{t+\Delta t}\right) \tag{28}$$

…

$$\left|\frac{d^{g-2}x}{dt^{g-2}}\right|_{t+\Delta t}=\left|\frac{d^{g-2}x}{dt^{g-2}}\right|_{t}+\Delta t\left(\left(1-\alpha_{g-2}\right)\left|\frac{d^{g-1}x}{dt^{g-1}}\right|_{t}+\alpha_{g-2}\left|\frac{d^{g-1}x}{dt^{g-1}}\right|_{t+\Delta t}\right) \tag{29}$$

$$\ddot{\boldsymbol{x}}(t+\Delta t)=\boldsymbol{f}(\boldsymbol{x}(t+\Delta t),\dot{\boldsymbol{x}}(t+\Delta t),t+\Delta t) \tag{30}$$

7 GENERAL CONSIDERATIONS

A drawback of these methods is that they require the initial values at up to the $g-1$ derivative. In some cases (see the pendulum example), analytic derivatives can be obtained. Another option is to obtain the derivatives in a finite differences approximation. Finally, one can start the algorithm taking those values as zero. This will harm precision, but in some cases that is not a major setback. In any case, one could probably start the algorithm with a very small timestep and afterwards increase it (this has not been tested by the authors yet).

A higher degree will usually mean better convergence, but will not increase the required amount of function evaluations. This is a great advantage for these methods, because in most problems, the bottleneck is the evaluation of the function to integrate. Obviously, in implicit methods the evaluation is usually performed iteratively, so convergence will play a major role in computational cost, but in explicit methods or in conditionally explicit methods in situations where they are explicit, this should be of a great advantage, because the integrator equations are simple vector operations (Level 1 BLAS).

The introduction of restrictions in Differential Algebraic Equations (DAEs) can be directly performed to the system of equations, although this does not alleviate by itself the stability issues that might appear in DAEs (see [6], among others).

8 EXAMPLES. ODEs

We have intentionally left the implicit methods out of these tests. This is due to the fact that the strong point of implicit methods are to solve stiff problems and, thus, it is not reasonable to test their efficiency against explicit methods in non stiff problems.

The computer used for the tests is a XeonE5645@2.4GHz. Times have been measures as averaged for 10 runs.

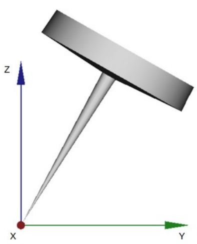
To check the efficiency of the methods, we have chosen the first example from IFTOMM Multibody benchmark. It is a simple pendulum of puntual mass 1 kg and a massless rod, 1m lenght. In this problem, one needs to take 10 seconds of simulation keeping the energy drift below 5e-5J. We refer the reader to the IFTOMM Multibody Benchmark for a description of the problem. An issue with the presented solutions in the benchmark is that they are solved using general purpose programs, which require restrictions to operate. An exception is the Opensim solution which uses a general purpose program, but it uses relative coordinates which, in this case, is identical to use minimal coordinates. But, in any case, it is a general purpose program, and thus, it has some additional work to be done. Thus, in order to be able to compare to a similar solver, we have included an RK4 solution, which should be quite competitive. The tested methods are both of 5th degree. One is explicit and the other conditionally explicit. Higher order derivatives for initial conditions in this case have been obtained with analytic expressions. We have chosen two solutions. The one asked in the problem and a high precision solution where the energy drift is limited to 5e-11 J. Results are presented in table 1.

Table 1. *Performance in the tests. Planar simple pendulum*

	Normal precision (max error: 5e-5J)			High Precision (max error: 5e-11J)		
Method	Time	Timestep	Error	Time	Timestep	Error
RK4	4.64431E-05	3.63e-2	4.98566e-05	7.20875E-04	1.81e-3	4.98632e-11
E5D	7.23016E-05	1.3e-2	4.93116e-05	2.15565E-03	4.33e-4	4.98037e-11
CE5D	3.6809e-05	2.49e-2	4.96226e-05	8.40650E-04	9.3e-4	4.98366e-11

Obviously all those ODE implementations outperform with a generous margin the results in the IFTOMM multibody benchmark, which requires 2.5 secs for the required solution and 0.637 secs for a result with a precision of 4e-11. But the real interest resides in the comparison of the explicit and conditionally explicit methods. The conditionally explicit algorithm (CE5D) outperforms RK4 by a small margin in normal precision and is outperformed in high precision. The explicit algorithm keeps the pace in normal precision and lags a bit in high precision, but not for a large margin. One should take into account that, in this case, due to the lack of gyroscopic effects, the conditionally explicit algorithm becomes explicit.

The next example is a 3D spinning top example. The spinning top starts with an angular velocity of $4\pi\,rad/s$ along its axis and an inclination with respect to the vertical axis of $\pi/6\,rad$. Total mass of the spinning top is 0.02 Kg and the moments of inertia respect to the principal inertia axix are of values 0, $0$and $2\text{e-}4\,kgm^2$. The distance from the pivot of the spinning top and the center of gravity is of 0.05 m. Total simulation time is of $10\,s$. Gravity is considered $9.81\,m/s^2$. The target is to perform the simulation keeping the energy drift below $1\text{e-}8\,J$.



$$\ddot{x}(t+\Delta t)=f(x(t+\Delta t),\dot{x}(t+\Delta t),t+\Delta t)$$

Figure 1. *Spinning top. Problem description and movement of the center of gravity in the xy plane (units are meters).*

This problem includes gyroscopic effects, but it is also a problem where energy is conservative. Thus, here the conditionally explicit method behaves as implicit (it still requires less computational cost in each iteration than a common implicit method such as Newmark). Results are shown in table 2. In this case higher order derivatives of initial conditions have been obtained numerically.

The conditionally explicit method pays the price of becoming implicit. In this problem the here presented explicit method takes the lead with a small margin with respect to the RK4 method.

Table 2. *Performance in the tests. Spinning top*

Method	Time	Timestep	Error
RK4	1.89218E-02	4e-3	9.12642e-09
E5D	1.57005E-02	1.4e-3	7.54016e-09
CE5D	8.13729E-02	1.2e-3	8.02416e-09

9 A DAE EXAMPLE

The real advantage of the structural methods rely on the application to DAEs. The fact that the restrictions can be directly applied to the problem leads to a very good performance of them. When compared to projection methods [7], restrictions are applied in all the function evaluations, and not only on the final result, thus improving convergence. When compared to Baumgarte [8], they allow for a more simple approach, without the need to tune parameters. We shall now show the results of the planar simple pendulum problem using a DAE formulation, solved using RK4 with projections and the conditionally explicit method. Results can be seen in table 2.

The conditionally explicit method is able to solve the problem quite faster than the RK4 to a similar precision. The effect of the unrestricted evaluations can be seen in the required timestep, which is in the conditionally explicit method even larger than the required in RK4. Here one must take into account that RK4 requires 4 evaluations by timestep and the conditionally explicit method only one. The results for the higher precision requirement are even better for the conditionally explicit method, which requires about four times less time than RK4. An important fact to take into account is that in this case higher order derivatives of initial conditions have all been set to 0.

Table 2. *Performance in the tests. Simple pendulum, DAE formulation*

	Normal precision			High precision		
Method	Time	Timestep	Error	Time	Timestep	Error
RK4	7.12920E-03	1.08e-2	4.83552e-05	2.2292E-01	3.44e-4	5.0326e-11
CE5D	2.79836E-03	1.24e-2	4.84126e-05	5.34952E-02	4.8e-4	4.95497e-11

10 CONCLUSIONS AND FUTURE WORK

Several alternative approaches for the integration of multibody dynamics have been presented. These methods seem to be able to deliver a performance comparable to that of Runge-Kutta methods, although still these are preliminary conclusions and a lot of testing has to be put on them. They have the drawback that they require initial conditions not only for position and velocity, but also for higher derivatives. This can be solved by a finite differences approximation. The great advantage of these methods is that they do not need to increase the amount of function evaluations to increase convergence, as happens with some multistep methods. This is a great feature, because not only should keep computational cost low, but also should eliminate the need to perform more evaluations in event driven simulations, when an event is triggered and leads to the need of a timestep change. It also should reduce overhead in variable timestep methods. Another feature of interest is that in Differential Algebraic Equations (DAEs) the restrictions can be added to the equilibrium equation and no need of projection methods or Baumgarte stabilization is required. Furthermore, in DAEs they seem to considerably reduce the cost. A couple of examples have also been presented. Future work includes further experimentation, stability and convergence analysis and a proper implementation for large problems.

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